JOURNAL OF AIRCRAFT
Vol. 44, No. 3, May–June 2007

Engineering Notes

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Surface Splines Generalization and Large Deflection Interpolation

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DOI: 10.2514/1.24571

I. Introduction

THE surface spline presented by Harder and Desmarais [1] ingeniously has been generally applied in aeroelastic analysis since the comment by Rodden et al. [2], and has been a standard method of function interpolation [3]. It is based on the small-deflection equation of an infinite plate, and so is also called infinite-plate spline (IPS). However, as a mathematical approach of data fitting, it is not limited to the hypothesis of small deflection. In this paper, we give a generalization of IPS to fit a vector value function with arbitrary number of variables, and apply it to the deformation interpolation of large deflection structures.

In [2], Rodden et al. presented the importance of interpolation methods concerning the coupling of aerodynamic/structure by quoting the words of Hitch, and reviewed the development of 2-D interpolation methods before 1970s. After that, there are two major progresses in the surface spline method. The first one is called the finite-surface spline by Appa [4], which is based upon the finite element method of a finite plate. It improves the extrapolation of the infinite-surface spline, but it is less applicable in aeroelastic analysis. The other is a kind of generalized IPS, named thin-plate spline (TPS), which extends the number of variables from two to three [5], and is applied in ZAERO [6].

The deformation of a very flexible structure has more than one component, and so a method to interpolate a vector function has to be established. In this note, the data fitting method between arbitrary dimension spaces is carried out by the further generalization of TPS. The transformation matrix defines a mapping from the original structure configuration to the set of displacement, or to the final configuration. Through the interpolation, the tangent mapping can be obtained as well to calculate the tangent or normal vectors of the configuration. This method provides a general interpolation scheme not limited to the structural and aerodynamic interface, but also applicable to interpolate any smooth data and curve/surface reconstruct.

Received 11 April 2006; revision received 18 January 2007; accepted for publication 5 February 2007.

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II. Mathematical Analysis

Consider a given vector set $\boldsymbol{X}_i = \{x_i^1, \ldots, x_i^N\}$ $(i = 1, 2, \ldots, n)$ in an N-dimensional space, and the corresponding image vectors $\boldsymbol{W}_i = \{w_i^1, \ldots, w_i^M\}$ $(i = 1, 2, \ldots, n)$ in an M-dimensional space. For the kth component of $\boldsymbol{W}(\boldsymbol{X})$, it can be fitted by TPS with N variables. So, the interpolation mapping could be written in matrix form,

$$\boldsymbol{W}(\boldsymbol{X}) = \boldsymbol{P}(\boldsymbol{X})\boldsymbol{W}_0 \tag{1}$$

where $\boldsymbol{P}$ is the transformation matrix of interpolation, which is defined only by the given vectors $\boldsymbol{X}_i$; the ith line of $\boldsymbol{W}_0$ is $\boldsymbol{W}_i$.

Denote the N-dimensional space that contains $\boldsymbol{X}_i$ as $\boldsymbol{X}$, and the M-dimensional space that contains $\boldsymbol{W}_i$ as $\boldsymbol{W}$, in an unconfused context. From Eq. (1), a differentiable mapping from $\boldsymbol{X}$ to $\boldsymbol{W}$ is established.

According to the definition of tangent vector and tangent mapping [7], if $\boldsymbol{X} \to^f \boldsymbol{W}$, then $\boldsymbol{TX} \to^{Tf} \boldsymbol{TW}$. Here, denote u as tangent vector of $\boldsymbol{X}$, v as tangent vector of $\boldsymbol{W}$, $\boldsymbol{DW}$ as tangent mapping of $\boldsymbol{W}(\boldsymbol{X})$, then,

$$\boldsymbol{v} = \boldsymbol{DWu} \tag{2}$$

In a Cartesian coordinate, $\boldsymbol{DW}$ is just the Jacobian of $\boldsymbol{W}(\boldsymbol{X})$, that is,

$$\begin{bmatrix} v^1 \\ \vdots \\ v^M \end{bmatrix} = \begin{bmatrix} \frac{\partial w^1}{\partial x^1} & \cdots & \frac{\partial w^1}{\partial x^N} \\ \vdots & \ddots & \vdots \\ \frac{\partial w^M}{\partial x^1} & \cdots & \frac{\partial w^M}{\partial x^N} \end{bmatrix} \begin{bmatrix} u^1 \\ \vdots \\ u^N \end{bmatrix} \tag{3}$$

III. Application in Aeroelasticity

A. Configuration Interpolation

In static aeroelastic analysis, using the undeformed and deformed configurations as interpolation entities is convenient. The configuration of structure is usually considered to be embedded in a 3-D space. The undeformed configuration could be 1-D, 2-D, or 3-D, and the deformed configuration is usually 3-D. From (1) and (2), the wanted grid set locations and tangent vectors of the deformed configuration are obtained, and the normal vectors of the aerodynamic grids and local attack angles can be calculated. Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ be two tangent vectors at a grid; the unit normal vector is then calculated by the cross product of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, that is,

$$\bar{\boldsymbol{n}} = \boldsymbol{v}_1 \times \boldsymbol{v}_2 / |\boldsymbol{v}_1 \times \boldsymbol{v}_2| \tag{4}$$

Denote $\bar{\boldsymbol{V}}$ as the unit vector of inflow air velocity in structure coordinate, and the local attack angle is α, then,

$$\sin\alpha = \bar{\boldsymbol{V}} \cdot \bar{\boldsymbol{n}} \tag{5}$$

B. Displacement Interpolation

For the large deflection problem, the displacement interpolation is an alternative approach in the aeroelastic analysis, but the calculation details are different from the configuration interpolation. In this case, the displacement of structure is given by (1), and the deformed configuration $\boldsymbol{Y}$ is

$$\mathbf{Y} = \mathbf{X} + \mathbf{W} \tag{6}$$

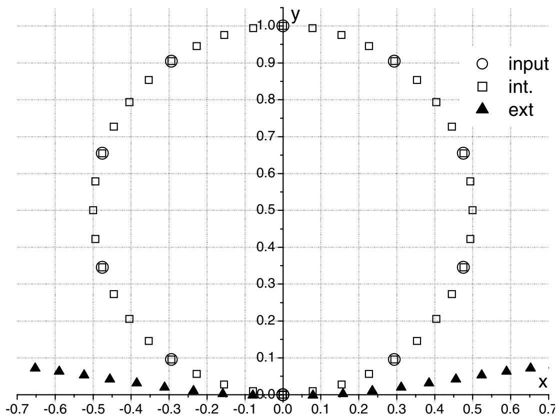

Fig. 1 Shape interpolating results include some extrapolations.

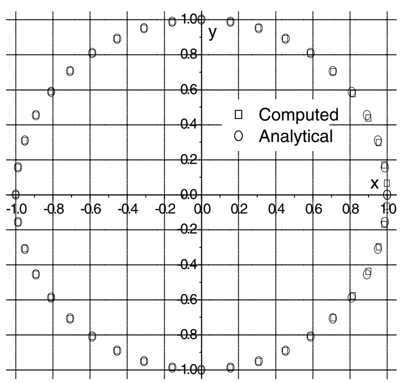

Fig. 2 Tangent vector results.

In principal, the dimension of $\boldsymbol{X}$ and $\boldsymbol{W}$ may be different. In (6), $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{W}$ are embedded in the structure space with three dimensions. So, the tangent mapping from the undeformed to the deformed configuration is modified as

$$\boldsymbol{v} = (\boldsymbol{I} + \boldsymbol{D}\boldsymbol{W})\boldsymbol{u} \tag{7}$$

where $\boldsymbol{I}$ is the identity matrix. The unit normal vector and the local attack angle are calculated by (4) and (5).

C. Force Interpolation

In aeroelastic analysis, the transformation between the aerodynamic and the structural force systems requires structure equivalence rather than static equivalence. Structure equivalence means that the two force systems deflect the structure equally [3]. When the aerodynamic forces $\boldsymbol{F}_a$ and their equivalent structure forces $\boldsymbol{F}_s$ do the same virtual work on their virtual deflections, respectively, the structure equivalence of two force systems is guaranteed:

$$\delta \boldsymbol{U}_a^T \boldsymbol{F}_a = \delta \boldsymbol{U}_s^T \boldsymbol{F}_s \tag{8}$$

where $\delta \boldsymbol{U}_a$ and $\delta \boldsymbol{U}_s$ are the virtual deflections, respectively, satisfying Eq. (1), that is, $\delta \boldsymbol{U}_a = \boldsymbol{P}(\boldsymbol{X})\delta \boldsymbol{U}_s$. So,

$$\boldsymbol{F}_s = \boldsymbol{P}(\boldsymbol{X})^T \boldsymbol{F}_a \tag{9}$$

IV. Numerical Example

Consider the in-plane deformation of a virtual beam with unit length, whose deformed configuration is a circle of unit diameter. The exact original and final shapes are

$$\boldsymbol{X}: x \in [0, 1]; \qquad \boldsymbol{W}: (w_1, w_2) = [\sin(2\pi x)/2, 1 - \cos(2\pi x)/2] \tag{10}$$

respectively. Eleven nodes are uniformly distributed on the origin line; their corresponding final locations are given to fit the circle. The interpolation coefficients are $h_i = 0$ and $\varepsilon = 0.1$. Define the error function of the shape interpolation as

$$\text{err}_w = |\boldsymbol{W} - \bar{\boldsymbol{W}}|/|\bar{\boldsymbol{W}} - (0, 0.5)| \tag{11}$$

Define the error function of the tangent vector as

$$\text{err}_v = |\boldsymbol{v} - \bar{\boldsymbol{v}}|/|\bar{\boldsymbol{v}}| \tag{12}$$

where $\boldsymbol{W}$ and $\boldsymbol{v}$ are the interpolation values; $\bar{\boldsymbol{W}}$ and $\bar{\boldsymbol{v}}$ are corresponding exact values.

Figures 1 and 2 plot the fitting circle and the tangent vectors at 41 grids with equal intervals. The interpolation precision is quite high, with location errors below 0.6%, and tangent vector errors below 6.5%. In Fig. 1, the output points are almost exact at the uniformly distributed points on the circle. In Fig. 2, the computed vectors have acceptable precision, but the vectors near edges are getting worse.

Figure 1 also plots some extrapolating points, which illustrate that the trustiness margin of extrapolation is quite small. The tangent vectors of extrapolating points are not plotted in Fig. 2, but Fig. 1 shows the tangent orientations trend to constant.

V. Conclusions

The generalized method based on the standard IPS is established for vector value function interpolation, and the numerical scheme for structure configuration characters is deducted through a simple differential geometry method. A 2-D example shows its availability and good precision. The method is easily applicable to the higher-dimensional problems. The general IPS can easily amalgamate into the frame of aeroelastic analysis to solve the structural and aerodynamic interface problems as standard IPS and TPS do. It can also be employed straightforwardly to interpolate any vector value functions, such as displacement, the gradient of scalar field, etc.

Acknowledgment

This work was supported by the National Natural Science Foundation (10432040, 10502003) and Supported Program for New Century Excellent Talents in University (NCET-04-0170).

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